Course 5: Mathematics in India: from Vedic period to modern times

1. **Introductory Overview**
   - Mahāvīrācārya on the all-pervasiveness of Gaṇita. The algorithmic approach of Indian Mathematics. Overview of development of Mathematics in India during the ancient and early classical Period (till 500 CE), later classical period (500-1250) medieval period (1250-1750) and the modern periods (1750- present). Proofs in Indian Mathematics. The genius of Srinivasa Ramanujan (1887-1920).Lessons from History.
2. **Mathematics in the Vedas and Śulva Sūtras**
   - Mathematical references in Vedas. The extant Śulbasūtra texts & their commentaries. The meaning of the word Śulbasūtra. Qualities of a Śulbakāra. Finding the cardinal directions. Methods for obtaining perpendicular bisector. Bodhāyana's method of constructing a square. The Bodhāyana Theorem (so called Pythagoras Theorem)
   - Applications of Bodhāyana Theorem. Constructing a square that is the difference of two squares. Transforming a rectangle into a square. To construct a square that is *n* times a given square. Transforming a square into a circle (approximately measure preserving). Rational approximation for √2. Construction of Citis. Details of fabrication of bricks, etc.
3. **Pāṇini's *Aṣṭādhyāyī***
   - Development of Vyākaraṇa or Śabadaśāstra. Pāṇini and Euclid. Method of Pāṇini's *Aṣṭādhyāyī*. Śiva-sūtras and Pratyāhāras. Context-sensitive rules and other techniques of *Aṣṭādhyāyī*. Pāṇini and zero. Patañjali on the method of *Aṣṭādhyāyī*. *Vākyapadīya* on *Aṣṭādhyāyī* as an upāya.
4. **Piṅgala's *Chandaḥśāstra***
   - Development of Prosody or Chandaḥśāstra. Long (guru) and short (laghu) syllables. Scanning of Varṇavṛtta and the eight Gaṇas. Pratyayas in Piṅgala's *Chandaḥśāstra*. Prastāra or enumeration in the form of an array. Saṅkhyā or the total number of metrical forms of *n* syllables. Naṣṭa and Uddiṣṭa (the association between a metrical form and the row-number in the prastāra through binary expansion). Lagakriyā or the number of metrical forms in the prastāra with a given number of Laghus. Varṇameru and the so called "Pascal Triangle".
5. **Mathematics in the Jaina Texts**
   - Place of Mathematics in Jaina literature. Important Jaina mathematical works. Jaina geometry. Circumference of a circle. Area of a circle. Relation between chord, śara (arrow) and diameter, etc. Approximation for the value of π. Notion of different types of infinity. The law of indices. Permutations and Combinations.
6. **Development of Place Value System**
   - Earliest evidence of the use of place value system. Numerals found in the inscriptions (Brāhmi & Kharoṣṭhi). Use of Zero as a symbol in Piṅgala's *Chandaḥśāstra*. References to use of decimal place value system in the commentary *Vyāsabhāṣya* on *Yogasūtra* and in Southeast Asian Inscriptions. Different systems of numeration employing place value system. Bhūtasaṅkhyā system. Āryabhaṭan system. Kaṭapayādi system. Algorithms for arithmetical operations based on decimal place value system.
7. ***Āryabhaṭīya* of Āryabhaṭa**
   - Āryabhaṭa, his period and his work *Āryabhaṭīya*. Names of the notational places. Square and Squaring. Algorithm for finding the square root. Cube and cubing. Algorithm for

finding the cube root. Formula for the area of a triangle. Bhāskara I on altitude and area of a triangle. Numerical examples

- Area of a circle, trapezium and other planar figures. Approximate value of π. Computation of tabular Rsines (geometric and difference equation methods). Approximate formula for Rsine (as given by Bhāskara I). Problems related to gnomonic shadow. Bhujā-koṭi-karṇa-nyāya, jyā-śara-nyāya and their applications. Arithmetic progressions. Finding sum of natural numbers, sum of sums, and so on.
- Some algebraic identities. Rule of three. Problems on interest calculation. Ekavarṇa-samikaraṇa and anekavarṇa-samikaraṇa. The Kuṭṭaka problem (sāgra and niragra-kuṭṭaka). Illustrative examples.

8. *Brāhmasphuṭasiddhānta* of Brahmagupta

- Introduction. Twenty logistics. Cube root. Rule of Three, Five Seven, etc. Mixtures. Interest calculations, etc. Progressions: Arithmetic and Geometric. Plane figures. Triangles, right triangles and quadrilaterals.
- Diagonals of a Cyclic quadrilateral. Rational triangles and quadrilaterals. Chords of a circle. Volumes with uniform and tapering cross-sections. Pyramids and frustum. Shadow problems.
- Mathematical operations with plus, minus and zero. Rules in handling surds (karaṇī) Operations with unknowns (avyakta-ṣaḍvidha). Equations with single unknowns (ekavarṇa-samīkaraṇa). Equations with multiple unknowns (anekavarṇa-samīkaraṇa). Equations with products of unknowns (bhāvita). Brahmagupta on kuṭṭaka. The Second order indeterminate equation (Vargaprakṛti). Bhāvanā principle and its applications.

9. Bakṣālī Manuscript

- The discovery of Bakṣālī Manuscript. Its antiquity and uniqueness. Use of symbols. Symbol for negative sign (kṣaya). Symbol for denoting unknown quantities (yāvatāvat). Solution of indeterminate equations. Formula for approximate value of surds. Some interesting problems involving simultaneous equations.

10. *Gaṇitasārasaṅgraha* of Mahāvīra

- Introduction. Arithmetical operations, operations with zero. Squares, cubes, square roots, cube roots. Arithmetical and Geometric progressions, Citi (summation). Manipulations with fractions and solutions of equations. Mixed problems including interest calculations.
- Vallīkāra-kuṭṭākara – linear indeterminate equations. Two and more simultaneous indeterminate equations. Other indeterminate equations. Vicitra-kuṭṭākara – Truthful and untruthful statements. Sums of progressions of various types. Variable velocity problem
- Plane figures**:** Circle**,** Dīrghavṛtta, Annulus. Ratio of circumference and diameter. Segment of a circle. Janya operations: rational triangles, quadrilaterals. Excavations: Uniform and tapering cross-sections, volume of a sphere. Time to fill a cistern. Shadow problems.

11. Development of Combinatorics

- Combinatorics in Āyurveda. Gandhayukti of Varāhamihira Mātrā-vṛttas or moric metres. Prastāra or enumeration of metres of *n*-mātrās in the form of an array. Saṅkhyā or the total number of metrical forms of given number of mātrās. The Virahāṅka sequence (so called Fibonacci sequence. Naṣṭa and Uddiṣṭa processes for finding the metrical form given the row-number and vice versa in a prastāra. Mātrā-meru to determine the number

of metrical forms with a given number of gurus. Representation of any number as a sum of Virahāṅka numbers.

- *Saṅgīta-ratnākara* of Śārṅgadeva (c.1225). Tāna-Prastāra or enumeration of permutations or tānas of svaras. Prastāra, the rule of enumeration of permutations in the form of an array. Khaṇḍameru and the processes of naṣṭa and uddiṣṭa. Factorial representation of Śārṅgadeva. Tāla-Prastāra: Enumeration of tāla forms. The tālāṅgas: Druta, Laghu, Guru and Pluta and their values. Prastāra: Rule of enumeration of all tāla-forms of a given value. Saṅkhyā and the Śārṅgadeva-sequence of numbers. The processes of naṣṭa and uddiṣṭa. Representation of natural numbers as sums of Śārṅgadeva-numbers. Laghu-Meru. The general relation between prastāra and representation of numbers.

12. *Līlāvatī* of Bhāskarācārya

- Introduction. Importance of *Līlāvatī*. Arithmetical operations: Inversion method, rule of supposition. Solution of quadratic equations. Mixtures. Combinations, progressions.
- Plane figures**:** Right triangles, applications. Sūcī problems. Construction of a quadrilateral: Discussion on earlier confusions. To find the second diagonal, given the four sides and a diagonal of a quadrilateral. Cyclic quadrilaterals. Value of π, area of a circle, surface area of a sphere, volume of a sphere
- Regular polygons inscribed in a circle. Expression for a chord in a circle. Excavations and contents of solids. Shadow problems (advanced problems). Importance of rule of proportions. Combinations (advanced problems).

13. *Bījagaṇita* of Bhāskarācārya

- Development of Bījagaṇita or Avyaktagaṇita (Algebra) and Bhāskara's treatise on it. Understanding of negative quantities. Development of algebraic notation. The Vargaprakṛti equation $X^2 - D Y^2 = K$, and Brahmagupta's bhāvanā process. The Cakravāla method of solution of Jayadeva and Bhāskara.
- Bhāskara's examples $X^2 – 61Y^2 = 1$, $X^2 – 67Y^2 = 1$. The equation $X^2 - D Y^2 = -1$. Solution of general quadratic indeterminate equations. Bhāskara's solution of a bi-quadratic equation.
- Review of the Cakravāla method. Analysis of the Cakravāla method by Krishnaswami Ayyangar. History of the solution of the "Pell's Equation" $X^2 - D Y^2 = 1$. Solution of "Pell's equation" by expansion of $\sqrt{D}$ into a simple continued fraction. Bhāskara semi-regular continued fraction expansion of $\sqrt{D}$. Optimality of the Cakravāla method.

14. ***Gaṇitakaumudī* of Nārāyaṇa Paṇḍita**

- Importance of *Gaṇitakaumudī*. Solutions of quadratic equations. Double equations of second and higher degree – rational solutions. Determinations pertaining to the mixture of things. Interest calculations – payment in installments
- Meeting of travelers. Progressions. Vārasaṅkalita: Sum of sums. The kth sum. The kth sum of a series in A.P. The Cow problem. Diagonals of a cyclic quadrilateral – Third diagonal, area of a cyclic quadrilateral. Construction of rational triangles with rational sides, perpendiculars, and segments whose sides differ by unity. Generalisation of binomial coefficients and generalized Fibonacci numbers.
- Vargaprakṛti. Nārāyaṇa's variant of Cakravāla algorithm. Solutions of Vargaprakṛti and approximation of square roots. Bhāgadāna: Nārāyaṇa's method of factorisation of numbers. Aṅkapāśa (Combinatorics). Enumeration (prastāra) of generalised mātrā-vṛttas (moric metres with more syllabic units in addition to Laghu and Guru). Some sequences

(paṅkti) and tabular figures (meru) used in combinatorics. Enumeration (prastāra) of permutations with repetitions. Enumeration (prastāra) of combinations.

15. Magic Squares

- The earliest textual references and references in inscriptions. The sarvatobhadra square of Varāhamihira. Nārāyaṇa's classification of magic squares into samagarbha (doubly-even numbers of the form 4m), viṣamagarbha (singly-even or numbersof the form 4m + 2) and viṣama (odd). Use of Kuṭṭaka to find the arithmetic sequences to be used in magic squares. 4x4 Pandiagonal magic squares of Nārāyaṇa.
- Ancient method for the construction of odd magic squares and doubly even squares. The folding method (samputīkaraṇa) of Nārāyaṇa for samagarbha squares. The folding method for Viṣama squares. Illustrative examples.

16. **Kerala School of Astronomy and Development of Calculus**

- Background to the Development of Calculus (c.500-1350). The notions of zero and infinity. Irrationals and iterative approximations. Second order differences and interpolation in computation of Rsines. Summation of infinite geometric series. Instantaneous velocity (tātkālika-gati). Surface area and volume of a sphere. Summations and Repeated summations (saṅkalita and vārasaṅkalita). The Kerala School of Astronomy and the Development of Calculus. Mādhava (c. 1340-1420) and his successors to Acyuta Piśāraṭi (c. 1550-1621). Nīlakaṇṭha (c.1450-1550) on the irrationality of π. Nīlakaṇṭha and the notion of the sum of infinite geometric series. Binomial series expansion. Estimating the sum $1^k + 2^k + ... n^k$ for large n.
- Mādhava Series for π. End-correction terms and Mādhava continued fraction. Transformed series for π which are rapidly convergent. History of Approximations to π. Nīlakaṇṭha's derivation of the Āryabhaṭa relation for second-order Rsine differences. Mādhava series for Rsine and Rcosine. Nīlakaṇṭha and Acyuta formulae for instantaneous velocity.
- Āryabhaṭa's sine table (makhi, bhaki, phaki…). Āryabhaṭa 's recursion relation and the approximation involved in it. Attempts to improve the sine values by Lalla, Govindasvāmi, Vaṭeśvara, etc. Bhāskara's formula for sin($A + B$) and its application. The refined recursion relation in *Taṅtrasangraha* and its commentary. Mādhava's sine series and the use of mnemonics vidvān, tunnabala etc. Mādhava's sine table. Comparison of sine-tables of Āryabhaṭa, Govindasvāmi, Vaṭeśvara and Mādhava.

17. Trigonometry and Spherical Trigonometry

- Crucial role of trigonometry in astronomy problems. Indian sines, cosines : Bhujājyā, Koṭijyā, sine tables. Interpolation formulae. Determination of the exact values of 24 sines. Bhāskara's Jyotpatti sin (18º), sin (36º).
- Sine of difference of two angles. Sines at the interval of 3º, 1.5º. Jīve-paraspara-nyāya. Sines at the interval of 1º. Trigonometry in later texts such as *Siddhāntatattvaviveka* of Kamalākara
- Spherical trigonometry in astronomy: Tripraśna problems. Applications to specific diurnal problems: Duration of day (carajyā), Time from shadow. Systematic treatment of spherical trigonometry problems in Nīlakaṇṭha's *Tantrasaṅgraha*. Proofs of *Tantrasaṅgraha* results in *Yuktibhāṣā.*

18. **Proofs in Indian Mathematics**

- Upapattis or proofs in Indian mathematical tradition. Early European scholars of Indian Mathematics were aware of upapattis. Some important commentaries which present upapattis. Bhāskarācārya II on the nature and purpose of upapatti. Upapatti of bhujā-koṭi-karṇa-nyāya (Baudhayana-Pythagoras theorem). Upapatti of kuṭṭaka process. Restricted use of tarka (proof by contradiction) in Indian Mathematics. The Contents of *Gaṇita-yukti-bhāṣā*. *Yukti-bhāṣā* demonstration of bhujā-koṭi-karṇa-nyāya. Estimating the circumference by successive doubling of circumscribing polygon.
- Expression for abādhās, area and circum-radius of a triangle. Theorem on the sum of the product of chords (jyāsaṃvarga-nyāya). Theorem on the difference of the squares of the chords (jyāvargāntara-nyāya). From jyāsaṃvarga-nyāya to jyotipatti (generation of tabular sines). The cyclic quadrilateral. Expression for the diagonals in terms of the sides. Expression for the area in terms of the diagonals. Expression for the area and circum-radius in terms of the sides.
- *Yuktibhāṣā* estimate of the samaghāta saṅkalita $1^k + 2^k + \ldots n^k$ for large n. *Yuktibhāṣā* estimate of Vārasaṅkalita. *Yuktibhāṣā* derivation of Mādhava Series for π. *Yuktibhāṣā* derivation of end-correction terms. *Yuktibhāṣā* derivation of Mādhava Rsine and Rcosine Series. Upapatti and "Proof". Lessons from history.

19. Mathematics in Modern India

- Continuing tradition of Indian Astronomy and Mathematics (1770-1870). Surveys of indigenous education in India (1825-1835). The Orientalist-Anglicist debate shaping the British policy on education (c.1835). Survival of indigenous education system till 1880. Modern Scholarship on Indian Mathematics and Astronomy (1700-1900). Rediscovering the Tradition (1850-1900). Development of Higher Education and Modern Mathematics in India (1850-1910). Srinivasa Ramanujan (1887-1920). Brief outline of the life and mathematical career of Ramanujan. Hardy's assessment of Ramanujan and his Mathematics (1922, 1940). Some highlights of the published work of Ramanujan and its impact. Selberg's assessment of Ramanujan's work (1988). The saga of Ramanujan's Notebooks. Ongoing work on Ramanujan's Notebooks. The enigma of Ramanujan's Mathematics. Ramanujan not a Newton but a Mādhava.
- Rediscovering the tradition (1900-1950). Rediscovering the tradition (1950-2010). Modern scholarship on Indian Mathematics (1900-2010). Development of modern mathematics in India (1910-1950). Development of modern mathematics in India (1950-2010). Development of higher education and scientific research in India (1900-1950). Development of higher education and scientific research in India (1950-2010). Comparison with global developments.

References

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2. C. N. Srinivasiengar, *History of Indian Mathematics*, The World Press, Calcutta 1967.
3. T. A. Saraswati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarsidass,

Varanasi 1979.
4. S. Balachandra Rao, *Indian Mathematics and Astronomy: Some Landmarks*, 3rd Ed. Bhavan's Gandhi Centre, Bangalore 2004.
5. G. G. Emch, M. D. Srinivas and R. Sridharan, Eds., *Contributions to the History of Mathematics in India*, Hindustan Book Agency, Delhi, 2005.
6. C. S. Seshadri, Ed., *Studies in History of Indian Mathematics*, Hindustan Book Agency, Delhi 2010.
7. G. G. Joseph, *Indian Mathematics Engaging the World from Ancient to Modern Times*, World Scientific, London 2016.
8. P. P. Divakaran, *The Mathematics of India Concepts Methods Connections*, Hindustan Book Agency 2018. Rep Springer New York 2018.
9. *Gaṇitayuktibhāṣā* (c.1530) of Jyeṣṭhadeva (in Malayalam), Ed. with Tr. by K. V. Sarma with Explanatory Notes by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, 2 Volumes, Hindustan Book Agency, Delhi 2008.